Linear Search with Bounded Resources

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A point is placed at random on the real line according to some known distribution F, and a search is made for this point, beginning at some starting points s on the line, and moving along the line according to some function $x(t)$. The objective of this article is to maximize the probability of finding the point while traveling at most d units. Characterizations of simple optimal searches are found for arbitrary distributions, for continuous distributions with continuous density everywhere (e.g., normal, Cauchy, triangular), and for continuous distributions with density which is continuous on its support (e.g., exponential, uniform). These optimal searches are also shown to be optimal for maximization of the expected number of points found if the points are placed on the line independently from a known distribution F.

1. INTRODUCTION

A point is placed on the real line R^1 according to a known probability distribution F, and then a search is made to determine the location of this point, by starting at some point s on R^1 and moving along the line. What is a good strategy for searching? Do optimal searches exist?

The work on search problems apparently stems from Bellman [2]; Bruss and Robertson [3] and Gal [6] have good reviews of the literature. Beck [1] uses the example of a man in an automobile who is looking for an object along a road; the automobile travels at a certain fixed speed and his objective (the same objective as that of Bellman, Gal, and Bruss and Robertson) is to minimize the expected time until the object is found.

The objective in this article, on the other hand, is to analyze a linear search problem with respect to maximizing the *probability* of finding the object, given that the searcher may travel only fixed finite distance. (In the automobile example, the bounded resources may perhaps be interpreted as "with a fixed amount of fuel," whereas in Beck's example good searches use arbitrarily large amounts of fuel.) The essential difference from the Bellman problem, of course, is the cost function and not just the bounded resources. The main results of this article include the existence and characterizations of simple optimal searches for general distributions, for continuous distributions with everywhere continuous densities (e.g., normal, Cauchy, triangular), for continuous distributions with densities which are continuous on their supports (e.g., uniform, exponential). They include also the proof that the optimal searches for a single point are

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optimal for maximization of the expected number of i.i.d. points found by searches which utilize information on the location of points in the region searched but do not look ahead.

2. OPTIMAL SIMPLE SEARCHES

There are certainly many different reasonable definitions of linear searches of given length, and the one chosen here seems to be the most general. Intuitively, a linear search of length d is any trajectory along the real line which moves a total of d units, including jumps and motion in both directions. Formally, a *linear search* is just an element of the set of functions BV [0, 1], the Banach space of real-valued functions on [0,1] which have bounded variation, with the total variation norm

$$\|x\| = \sup \Sigma \, |x(t_i) - x(t_{i-1})|,$$

where the supremum extends over all partitions of [0, 1], by finite collections of points $0 = t_0 < \cdots < t_n = 1$. Thus the function $x(\cdot)$ represents the search, $x(t)$ is the position of the searcher at time t, and $\|x\|$ is the total length of the search. The set $S(d)$ of x in BV [0, 1] with $\|x\| \leq d$, where $d > 0$, will denote the linear searches with fixed resources d, and the searches with resources d which start at $s \in \mathbb{R}^1$ are those $x \in S(d)$ for which $x(0) = s$. The maximum distance that a search x in $S(d)$ can travel from its starting point is d, as is easily seen from the definition of $\|x\|$. If $\mathscr{R}(x)$ denotes the range of the function x, and co(A) the closed convex hull of the set A, then the probability that the object is located by the search x is

$$\Lambda_F(x) = \mu(\text{co}(\mathscr{R}(x))),$$

where μ is the Borel measure on $\mathbb{R}^1$ generated by the probability distribution F. The tacit assumption here is that if the searcher travels from point A to point B, then he has visited or "seen" every point in between, whether or not the transition from A to B was instantaneous or continuous. The crucial difference between Beck's problem and this one is that the searches in this article are of finite length. The interval [0, 1] in BV[0, 1] can be thought of as "time"; clearly any other closed interval will serve the same purpose, and the unit interval was simply chosen for convenience.

Next, define

$$\Lambda^*_{F,d} = \sup\{\Lambda_F(x)\colon x \in S(d)\},$$

and

$$\Lambda^*_{F,d,s} = \sup\{\Lambda_F(x)\colon x \in S(d),\, x(0) = s\}.$$

Intuitively, $\Lambda^*_{F,d}$ represents the highest possible probability of finding the object if the searcher is allowed to start anywhere on $\mathbb{R}^1$, but can travel no more than d units, while $\Lambda^*_{F,d,s}$ represents the optimal probability of finding the object given starting position s and d units of travel. Clearly $\Lambda^*_{F,d} \geq \Lambda^*_{F,d,s}$ (and in fact $\Lambda^*_{F,d} = \max\{\Lambda^*_{F,d,s}: s \in \mathbb{R}\}$; see Theorem 2.2 below).

Let $Q_{F,d}$ denote the Lévy concentration function for closed intervals of length d (cf. [7]), i.e., $Q_{F,d} = \sup\{\mu[a, a + d]: a \in \mathbb{R}\}$.

THEOREM 2.1: $\Lambda^*_{F,d} = Q_{F,d}$.

PROOF: Since for any $x \in S(d)$, $\text{co}(\mathscr{R}(x))$ is an interval of length no more than d, $\Lambda_F(x) \leq Q_{F,d}$ by definition of $Q_{F,d}$. Therefore, $\Lambda^*_{F,d} \leq Q_{F,d}$. On the other hand, let $\varepsilon > 0$ be arbitrary and $[a, a + d]$ satisfy $\mu([a, a + d]) > Q_{F,d} - \varepsilon$. Since the function $x(t) = a + td$, $t \in [0, 1]$ is in $S(d)$,

$$\Lambda^*_{F,d} \geq \Lambda_F(x) = \mu([a, a + d]) > Q_{F,d} - \varepsilon.$$

□

The objective is to find *optimal* searches, that is, searches in the appropriate class (i.e., with fixed length and either an arbitrary or fixed starting point) which have the highest probability of finding the hidden object. Formally speaking, an optimal search is as follows.

DEFINITION: The search x^* is (F, d)*-optimal* if $x^* \in S(d)$ and $\Lambda^*_{F,d} = \Lambda_F(x^*)$. The search x^* is (F, d, s) *optimal* if $x^* \in S(d)$, $x^*(0) = s$, and $\Lambda^*_{F,d,s} = \Lambda_F(x^*)$.

REMARK: A well-known inequality of Lévy (cf. [7], p. 27) giving a sharp lower bound for the concentration function in terms of the variance (and vice versa) may be translated, via Theorem 2.1, into a sharp bound for the search probabilities $\Lambda^*_{F,d}$ in terms of the variance. Reference [4] contains generalizations of Lévy's inequalities which can be used to bound $\Lambda^*_{F,d}$ and $\Lambda^*_{F,d,s}$.

THEOREM 2.2: For every probability distribution F, every $d > 0$, and every $s \in \mathbb{R}^1$, (F, d)- and (F, d, s)-optimal searches exist.

PROOF: The existence of (F, d)-optimal searches follows immediately from Theorem 2.1 above and Theorem 1.1.8 of Hengartner and Theodorescu [7], since for any F and $d > 0$, $Q_{F,d}$ is attained for some interval $[a, a + d]$. The existence of (F, d, s)-optimal searches will follow from Theorem 2.6. □

REMARK: A similar existence result, in a more general objective setting, is Theorem 2.1 of Fristedt and Heath [5].

The following theorem is another consequence of Theorem 2.1 and a well-known result of Lévy for concentration functions (cf. [7], Theorem 1.2.3).

THEOREM 2.3: The set of real numbers which are starting points for (F, d)-optimal searches is closed and nonempty.

DEFINITION: A search x is *simple* if there is a $\gamma \in [0, 1]$ so that x is linear on $[0, \gamma]$ and on $[\gamma, 1]$ and has a strict extremum at γ. If $\gamma \in (0, 1)$ then $x(\gamma)$ is called a *turning point* of x.

Informally, a search is simple if it moves in one direction with constant velocity until it reaches some point t, and then it either stops or moves back in the opposite direction with constant velocity. The next task will be to show that optimal simple searches always exist.

LEMMA 2.4: Fix $d > 0$ and $s \in \mathbb{R}^1$. Given $\varepsilon > 0$ there is a simple search $x \in S(d)$ starting at s such that

$$\Lambda_F(x) > \Lambda^*_{F,d,s} - \varepsilon.$$

PROOF: Let $y \in S(d)$ be such that $y(0) = s$ and $\Lambda_F(y) > \Lambda^*_{F,d,s} - \varepsilon$. Let $[a, b]$ be the closed convex hull of $\mathscr{R}(y)$. Then $\Lambda_F(y) = \mu([a, b])$. Let $t_n \in [0, 1]$ be such that $y(t_n) \to a$ and $r_n \in [0, 1]$ be such that $y(r_n) \to b$. Then

$$d \geq \|y\| \geq |s - y(t_n)| + |y(t_n) - y(r_n)| + |y(r_n) - y(1)|, \qquad \text{if } t_n \leq r_n,$$

or this inequality holds with the roles of t_n and r_n reversed if $t_n \geq r_n$. In any case, letting $n \to \infty$ shows that either $\|y\| \geq |s - b| + |b - a|$ or $\|y\| \geq |s - a| + |b - a|$. Assuming, without loss of generality, that the former holds and observing that $b \geq s$, define the simple search $x_0 \in S(d)$ starting at s by

$$x_0(t) = \begin{cases} s + 2(b - s)t, & 0 \leq t \leq 1/2, \\ b + 2(a - b)(t - 1/2), & 1/2 \leq t \leq 1. \end{cases}$$

Then $\|x_0\| = |s - b| + |b - a| \leq d$ and

$$\Lambda_F(x_0) = \mu([a, b]) = \Lambda_F(y) > \Lambda^*_{F,d,s} - \varepsilon. \qquad \square$$

DEFINITION: For a Borel probability measure μ and every real s and t let

$$P_s(t) = \begin{cases} \mu([2t - d - s, t]), & \text{if } t > s, \\ \mu([t, d - s + 2t]), & \text{if } t < s, \\ \mu([s, s + d]) \vee \mu([s - d, s]), & \text{if } t = s. \end{cases}$$

In other words, $P_s(t)$ is just the probability of finding the object with a simple search starting at s and turning back at t (if $s = t$, it represents the better of the two cases moving left d units, or moving right d units).

LEMMA 2.5: $\Lambda^*_{F,d,s} = \sup\{P_s(t): t \in [s - d/3, s + d/3]\}$.

PROOF: It is easily verified that for any simple search x, with $\|x\| = d$, and with a turning point $t = x(\gamma) \in [s - d/2, s + d/2]$, $\Lambda_F(x) = P_s(t)$. If x is simple but has no turning point then $\Lambda_F(x)$ is either $\mu([s, s + d])$ or $\mu([s - d, s])$. Lemma 2.4 shows that $\Lambda^*_{F,d,s}$ is the supremum of $\Lambda_F(x)$ over the set of simple searches in $S(d)$ starting at s. It is clear that one can restrict the supremum to those simple searches in $S(d)$ for which $\|x\| = d$. Therefore, the statement of Lemma 2.5 with $d/2$ rather than $d/3$ will be proven if it can be shown that for every simple $x \in S(d)$ starting at s, whose turning point is in $(s - d, s - d/2] \cup [s + d/2, s + d)$, there is a simple $x^* \in S(d)$ starting at s for which $\Lambda_F(x^*) \geq \Lambda_F(x)$.

Assume first that x has a turning point at γ with $x(\gamma) \in [s + d/2, s + d)$. Then for all $t \in [\gamma, 1]$, $s \leq x(t) < s + d$, for otherwise $\|x\| > d$. Therefore, if x^* is the simple search with no turning point which goes from s to $s + d$ then $\Lambda(x^*) = \mu([s, s + d]) \geq \mu([s, x(\gamma)]) = \Lambda(x)$. The same argument applies to $x(\gamma) \in (s - d, s - d/2]$. To complete the argument, observe that if the turning point lies in $(s + d/3, s + d/2)$ or $(s - d/2, s - d/3)$, a better strategy is the simple search with turning point at the other strategy's terminal point. □

The next result facilitates the identification of optimal searches by vastly reducing the class in which optimal searches are known to exist (for applications see Section 3).

THEOREM 2.6: There is always an (F, d, s)-optimal simple search of length d which either has no turning points, or has exactly one turning point located not more than $d/3$ units from the starting point s.

PROOF: If the function $P_s(t)$ can be shown to be upper semicontinuous (usc) on $[s - d/3, s + d/3]$ then the theorem will follow, for then $P_s(t)$ achieves its maximum on that interval.

Let $t_n \to t_0 \in (s, s + d/3]$ and choose a subsequence $t_{n'}$, for which $P_s(t_{n'}) \to \limsup P_s(t_n)$. Then $[2t_0 - d - s, t_0] \supseteq C = \lim[2t_{n'} - d - s, t_{n'}]$ so that

$$\limsup P_s(t_n) = \lim P_s(t_{n'}) = P(C) \leq P_s(t_0).$$

The same argument shows that $P_s(t)$ is usc on $[s - d/3, s)$. At s one can take a further subsequence to get all the intervals of the same type and utilize the same argument. The maximum in the definition of $P_s(t)$ at $s = t$ assures the necessary inequality and completes the proof of the theorem. □

The next idea, that of a "density-doubling" point for a distribution, is a key step in the analysis of many search problems based on a distribution having a continuous density, as is seen in the following theorem.

DEFINITION: A turning point t of a simple search x is *density doubling* for the density f if

$$f(t) = 2f(x(1)).$$

THEOREM 2.7: If F has an everywhere continuous density, then every (F, d, s)-optimal single search of length d either has no turning point or has a turning point which is density doubling.

PROOF: Since f is continuous, it is an elementary fact that for every $t \in (s - d/2, s + d/2) - \{s\}$ the function $P_s(t)$ is differentiable. If $t > s$ then $P_s'(t) = 2f(d - s + 2t) - f(t)$. We know by Theorem 2.6 that a maximum is attained either at $t = s$ or in $(s - d/2, s + d/2) - \{s\}$ and if the latter set, there must be a critical point. Setting $P_s'(t)$ to 0 and observing that the maximum at $t = s$ entails no turns proves the claim. □

Let S be the support of F and ∂S be its boundary.

THEOREM 2.8: If, on the interior of S, F has a continuous density with respect to Lebesgue measure, then there is an (F, d, s)-optimal simple search of length d which has either no turning point, a turning point or terminal point on ∂S, or a density-doubling turning point with both turning point and terminal point in the interior of S.

PROOF: From Theorem 2.6, it suffices to prove that if there are no optimal simple searches of length d without a turning point, then there are optimal simple searches of length d which fall into one or the other of the two remaining categories.

If an (F, d, s)-optimal simple search of length d has both its turning point and terminal point in the complement of ∂S, then the turning point t_0 must be density doubling. Here the density f is taken to be zero on the complement of S. If both points are in the complement of S then one can clearly move the turning point until it, or the terminal point, is on ∂S without changing the value of $\Lambda_F(x)$. Finally, searches with one point in the interior of S and the other in the complement of S cannot be optimal for one can clearly increase $\Lambda_F(x)$ by a slight perturbation of the turning point. □

3. OPTIMAL SEARCH STRATEGIES FOR SEVERAL CLASSICAL DISTRIBUTIONS

Theorems 2.6–2.8 may be used to greatly facilitate identification of optimal search strategies for many common distributions, and the purpose of this section is to record several such applications. For the standard normal distribution, by using the density-doubling characterization (Theorem 2.7) and somewhat involved but elementary calculations, one can prove the following.

THEOREM 3.1: An optimal search for $N(0, 1)$ entering at $s \leq 0$ with resources d goes right to $s + d$ if $d(d + 2s) \leq 2 \ln 2$ and otherwise goes left to $-(2/3)(d - s) + (1/3)\sqrt{(d - s)^2 + 6 \ln 2}$ and then right using the remainder of the resources.

Some special cases seem noteworthy. If $s = 0$, then for $d^2 > 2 \ln 2$ first one goes left (or right of course) then turns right. But if $d^2 \leq 2 \ln 2$, then there are no turns. If s is arbitrary, $s \leq 0$, then as d becomes large the optimal strategy is roughly to turn left first, using approximately one third of the resources to $s - d/3$, and then head right to end at $s + d/3$.

The optimal search for the standard normal is easily translated into an optimal search for an arbitrary normal. In fact, this holds for all scale-location transformations of an arbitrary F. Let the distribution be $F(v)$ and denote, here only, an optimal simple search for F with initial resources d and starting point s by $x(0, 1, s, d, t)$ for $0 \le t \le 1$. Then one can show that for the scale-location version $F[(v - a)/b]$, an optimal search strategy for a start at s and with initial resources d is

$$x(a, b, s, d, t) = a + bx\left(0, 1, \frac{s-a}{b}, \frac{d}{b}, t\right)$$

for $0 \le t \le 1$.

Using Theorem 2.7 one can also prove the following two theorems.

THEOREM 3.2: For the Cauchy distribution whose density is

$$f(x) = \frac{1}{\pi(1 + x^2)},$$

if $s \le 0$ and $d \le d_0 = -s + \sqrt{2s^2 + 1}$ then the simple search which has no turns and proceeds to the right to $s + d$ is optimal. If $d > d_0$ then the simple search which first goes left to

$$s - d + \frac{\sqrt{2}}{2}\sqrt{1 + (d - s)^2},$$

and then proceeds right is optimal.

THEOREM 3.3: For the triangular distribution on $[-a, a]$ whose probability density is

$$f(x) = \begin{cases} \frac{1}{a}\left(1 + \frac{-|x|}{a}\right), & \text{for } |x| \le a, \\ 0, & \text{otherwise,} \end{cases}$$

if entry is at $s \in [-a, 0]$ then for $d \le d_0 = (a - 3s)/2$ the search which proceeds to the right with no turns to $s + d$ is optimal. If $3a + s \ge d > d_0$ then the simple search of length d which first proceeds left to

$$t = \frac{2s - 2d + a}{5},$$

then right, is optimal. If $d > 3a + s$, then the search which proceeds to $-a$ then right to a is optimal.

For the uniform and exponential distributions, Theorem 2.8 yields the following results.

THEOREM 3.4: If F is the uniform distribution on [0, 1], and $s \leq 1/2$, then an optimal strategy is for $d \leq 1 - s$ go straight right to $s + d$, and for $d > 1 - s$, turn at $(s - d + 1)/2$ and terminate at one.

EXAMPLE 3.5: If F is the exponential distribution with mean 1, $s \geq 0$, an optimal strategy is if (s, d) is in Region A use (d), in B use (e), in C use (e), in D use (c), in E use (e), in F use (a), in G use (e), and in H use (b) where the regions are shown in Figure 1 and strategies (a), (b), (c), (d) and (e) are the simple searches described by

(a) straight right terminating at $s + d$,
(b) straight left terminating at $s - d$,
(c) a density doubling turning point in $[(s - d/2)^+, s]$,
(d) a turning point at zero and termination point $d - s$,
(e) a turning point at $(d + s)/2$ and termination point 0.

REMARKS: The solid lines in Figure 1 were derived analytically while the dotted line was computed numerically.

The following facts are useful in showing that the above strategy is optimal for the exponential distribution. To have a density-doubling point, the distance between the turning point and the termination point must be log 2. A turning point that is farther than $d/3$ from s is suboptimal. Combining these two ob-

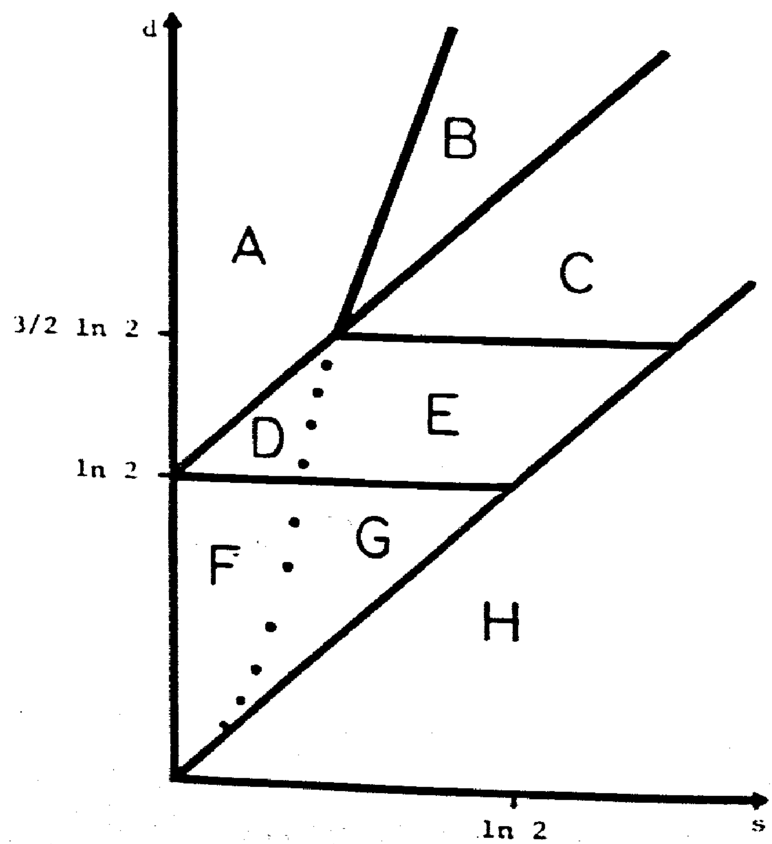

Figure 1.

servations yields the result that density doubling points can only be optimal when $\log 2 \leq s \leq (s + \log 2) \wedge (3/2) \log 2$.

Between the strategies (b) and (e) it is easy to show that (b) is better than (e) if $d < s$, and (e) is better than (b) if $d > s$. Among the strategies (a), (c), and (d), (a) is optimal if $d < \log 2$, (c) is optimal if $\log 2 < d < (s + \log 2) \wedge (3/2) \log 2$, and (d) is optimal if $d > s + \log 2$. [We can ignore the interval $((3/2) \log 2, s + \log 2)$ since the strategy with turning point at $s - d/2$ dominates (a), (c), and (d) but is dominated by (b) and (e).]

Thus, we need only compare the best of (b) and (e) with the best of (a), (c), and (d). In most cases this is easily resolved, frequently by noting that turning points farther than $d/3$ units from s are suboptimal. For example, in Region B, we must compare (e) and (d). Since $d < 3s$, (e) is optimal while in Region A, $d > 3s$, and (d) is optimal. The only regions that are difficult to compare analytically are D ∪ E and F ∪ G. In D ∪ E we numerically compared (e) and (c) and found that (c) is optimal in D and (e) is optimal in E. Similarly, we numerically compared (e) and (a) in F ∪ G.

4. LINEAR SEARCH FOR n POINTS

Suppose that n points $X_1, \ldots, X_n$ are placed independently at random on the real line according to a known probability distribution function. Let N_x denote the number of points encountered by the search x. Consider the problem of selecting a search x (with fixed starting point and fixed length) that maximizes $E[N_x]$, the expected number of points encountered. If $n = 1$, this problem is identical to the problem of the first three sections where we only needed to consider searches $x(t)$, $t \in [0, 1]$. For $n > 1$, we allow the search to be a function of $X_1, \ldots, X_n$, and let $x(t, X_1, \ldots, X_n)$ denote the position of the searcher at time t. Of course, we need to ensure that the searcher at time t only uses information discovered up to time t. Searches that satisfy this property as well as starting at s and traveling at most d units will be called (s, d)-*feasible*. We will delay the formal description until after stating the results.

THEOREM 4.1: Let x be an (s, d)-feasible search and let $X_1, \ldots, X_n$ be independent random variables with X_i having distribution F_i. Then

$$E[N_x] \leq \sum_{i=1}^{n} \Lambda^*_{F_i,d,s}.$$

COROLLARY 4.2: Suppose $X_1, \ldots, X_n$ are independent, identically distributed random variables with distribution F. Any search starting at s of length d which maximizes the probability of finding a single point with distribution F is also optimal for maximizing the expected number of points in the n-point problem.

Formally, the trajectories of an (s, d)-feasible search $x(t, X_1, \ldots, X_n)$, $t \in [0, 1]$, are sample paths of a separable stochastic process satisfying

(i) the initial point is s; i.e., $x(0, X_1, \ldots, X_n) = s$;
(ii) the search travels at most d units; i.e., $x(\cdot, X_1, \ldots, X_n) \in S(d)$; and

(iii) the searcher only uses information at time t which has been discovered up to time t. Hence, if X_i is not in J_t, the region searched during $[0, t]$, then the search during $[0, t]$ would be unchanged if X_i were located at any other point not in J_t. That is, if $X_i \notin J_t$, then for every $w \notin J_t$ and $u \le t$

$$x(u, X_1, \ldots, X_{i-1}, w, X_{i+1}, \ldots, X_n) = x(u, X_1, \ldots, X_{i-1}, X_i, X_{i+1}, \ldots, X_n). \quad (1)$$

Note that with these properties, the searcher even learns the *identity* of an encountered point, i.e., whether the point was X_1 or X_2 or . . . , which itself may contain useful information.

PROOF (of Theorem 4.1): The number of points encountered N_x can be written as $N_x = \sum_{i=1}^{n} I_{J_1}(X_i)$ where J_1 is the random interval generated by the search x and I_B is the indicator function of the set B. (Since the process is separable, it is straightforward to show that $I_{J_1}(X_i)$ is measurable.) Hence

$$EN_x = \sum_{i=1}^{n} \Pr\{X_i \in J_1\}.$$

Suppose we consider a different objective function: maximize $\Pr\{X_i \in J_1\}$ for a fixed index i over feasible searches. Furthermore, suppose that instead of gathering information about $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n$ during the search, we are given $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n$ and allowed to use this information. In other words, the third property of a feasible search (1) is only required for the i^{th} component instead of all components. Of course, we cannot do worse with this extra information. But now we have the problem of searching for a single point with distribution $F_{i|\bar{i}}$, where $F_{i|\bar{i}}$ is the conditional distribution of X_i given $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n$. From Section 2, the maximum probability of finding X_i is $\Lambda^*_{F_{i|\bar{i}},d,s}$. Combining this with (1) yields

$$EN_x \le \sum_{i=1}^{n} E[\Lambda^*_{F_{i|\bar{i}},d,s}] = \sum_{i=1}^{n} E[\Lambda^*_{F_i,d,s}] = \sum_{i=1}^{n} \Lambda^*_{F_{\bar{i}},d,s},$$

where the first equality followed from mutual independence of $X_1, \ldots, X_n$. □

PROOF (of Corollary 4.2): If there is a feasible search x which has

$$\Pr\{X_i \in J_1\} = \Lambda^*_{F_i,d,s}$$

for every i, $i = 1, \ldots, n$, then by Theorem 4.1, x maximizes the expected number points encountered. In particular, if $X_1, \ldots, X_n$ are i.i.d. this is satisfied, since the optimal simple search for one point yields a nonrandom set J_1 and $\Pr\{X_i \in J_1\} = \Lambda^*_{F,d,s}$ for $i = 1, \ldots, n$. □.

REMARK: For the situation where the searcher may select the starting point, the analogs of both Theorem 4.1 and Corollary 4.2 [e.g., $E(N_x) \leq \Sigma_{i=1}^{n} \Lambda_{F_i,d}^{*}$] follow by the same argument.

Without independence, the conclusion of Corollary 4.2 may fail, as is illustrated by following example involving the search for two points located according to two exchangeable random variables.

EXAMPLE 4.7: Let X_1 be uniformly distributed on the interval $[-1, 1]$ and $X_2 = -X_1$. One optimal search in $S(1)$, starting at $s = 0$, for finding a single point in $[-1, 1]$ proceeds straight to the right. As a search for the two points X_1 and X_2 it will always uncover precisely one point, so the expected number found is 1. However, the search which proceeds to the right and reverses whenever it encounters a point, finds precisely one point with probability 2/3 and precisely two points with probability 1/3. The expected number found is $4/3 > 1$.

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