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BENFORD LAW, *significant-digit law*, *first-digit law* – A **probability distribution** on the significant digits of real numbers named after one of the early researchers, [1]. Letting $\{D_n\}_{n=1}^{\infty}$ denote the (base-10) *significant digit functions* (on $\mathbf{R}\backslash\{0\}$), i.e.,

$$D_n(x) = n\text{th significant digit of } x$$

(so, e.g., $D_1(0.0304) = D_1(304) = 3$, $D_2(0.0304) = 0$, etc.), Benford's law is the logarithmic probability distribution P given by

1) (*first digit law*)

$$\mathsf{P}(D_1 = d) = \log_{10}(1 + d^{-1}), \quad d = 1, \dots, 9;$$

2) (*second digit law*)

$$\mathsf{P}(D_2 = d) = \sum_{k=1}^{9} \log_{10}\left(1 + (10k + d)^{-1}\right),$$

$$d = 0, \dots, 9$$

3) (*general digit law*)

$$\mathsf{P}(D_1 = d_1, \dots, D_k = d_k) =$$

$$= \log_{10}\left[1 + \left(\sum_{i=1}^{k} d_i \cdot 10^{k-i}\right)^{-1}\right]$$

for all $k \in \mathbf{N}$, $d_1 \in \{1, \dots, 9\}$ and $d_j \in \{0, \dots, 9\}$, $j = 2, \dots, k$.

An alternate form of the general law 3) is

4) $\mathsf{P}(\text{mantissa} \le t/10) = \log_{10} t$ for all $t \in [1, 10)$.

Here, the *mantissa* (base 10) of a positive real number x is the real number $r \in [1/10, 1)$ with $x = r \cdot 10^n$ for some $n \in \mathbf{Z}$; e.g., the mantissas of both 304 and 0.0304 are 0.304.

More formally, the logarithmic **probability measure** P in 1)–4) is defined on the **measurable space** $(\mathbf{R}^+, \mathcal{M})$, where $\mathbf{R}^+$ is the set of positive real numbers and $\mathcal{M}$ is the (base-10) *mantissa sigma algebra*, i.e., the sub-sigma-algebra of the Borel σ-algebra generated by the significant digit functions $\{D_n\}_{n=1}^{\infty}$ (or, equivalently, generated by the single function $x \mapsto \text{mantissa}(x)$). In some combinatorial and number-theoretic treatises of Benford's law, $\mathbf{R}^+$ is replaced by $\mathbf{N}$, and P by a finitely-additive **probability measure** defined on all subsets of $\mathbf{N}$.

Empirical evidence of Benford's law in numerical data has appeared in a wide variety of contexts, including tables of physical constants, newspaper articles and almanacs, scientific computations, and many areas of accounting and demographic data (see [1], [5], [6], [7]), and these observations have led to many mathematical derivations based on combinatorial (e.g., [2]), analytic ([3], [8]), and various urn-scheme arguments, among others (see [7] for a review of these ideas).

Benford's law P can also be characterized by several invariance properties, such as the following two. Say that a probability measure $\widehat{\mathsf{P}}$ on the *mantissa space* $(\mathbf{R}^+, \mathcal{M})$ is *scale-invariant* if $\widehat{\mathsf{P}}(sS) = \widehat{\mathsf{P}}(S)$ for every $S \in \mathcal{M}$ and $s > 0$, and is *base-invariant* if $\widehat{\mathsf{P}}(S^{1/n}) = \widehat{\mathsf{P}}(S)$ for every

$S \in \mathcal{M}$ and $n \in \mathbf{N}$. Letting P denote the logarithmic probability distribution given in 1)–4), then (see [4])

- P is the unique probability on $(\mathbf{R}^+, \mathcal{M})$ which is scale-invariant;
- P is the unique atomless probability on $(\mathbf{R}^+, \mathcal{M})$ which is base-invariant.

A statistical derivation of Benford's law in the form of a central limit-like theorem (cf., e.g., **Central limit theorem**) characterizes P as the unique limit of the significant-digit frequencies of a sequence of random variables generated as follows. First, pick probability distributions at random, and then take random samples (independent, identically distributed random variables) from each of these distributions. If the overall process is scale- or base-neutral (see [5]), the frequencies of occurrence of the significant digits approach the Benford frequencies 1)–4) in the limit almost surely (i.e., with probability one; cf. also **Convergence, almost-certain**).

There is nothing special about the decimal base in 1)–4), and the analogue of Benford's law 4) for general bases $b > 1$ is simply

$$\text{Prob}\left(\text{mantissa}\left(\text{base}\, b\right) \leq \frac{t}{b}\right) = \log_b t$$

for all $t \in [1, b)$.

Applications of Benford's law have been given to design of computers, mathematical modelling, and detection of fraud in accounting data (see [5], [7]).

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BENJAMIN–FEIR INSTABILITY – In his 1847 paper, G.G. Stokes proposed the existence of periodic wave-trains in non-linear systems. In the case of waves on deep water, the first two terms in the asymptotic expansion employed by Stokes are given by

$$\eta(x, t) = a\cos(\zeta) + \frac{1}{2}ka^2\cos(2\zeta),$$

where $\zeta = kx - \omega t$ and $\omega^2 = gk(1 + k^2a^2)$. Not everyone was convinced that the series converges and therefore that periodic waves actually exist. Convergence of the series for waves on infinitely deep water was finally established in 1925 by T. Levi-Civita and extended the next year to waves on water of finite depth by D.J. Struik (a brief history is given by [4]). With the existence of the *Stokes wave* established, it is quite remarkable that no one noticed its instability, until T.B. Benjamin and J.E. Feir during the 1960s (see, e.g., [4], [5]).

Adding perturbations with frequencies close to the *carrier frequency* ω, of the form

$$\epsilon(x, t) = \epsilon_+ \exp(\Omega t)\cos[kx - \omega(1 + \delta)t] + \\ + \epsilon_- \exp(\Omega t)\cos[kx - \omega(1 - \delta)t],$$

Benjamin and Feir used a linearized analysis to show that

$$\Omega = \frac{1}{2}\delta(\sqrt{2k^2a^2 - \delta^2})\omega.$$

Thus, the perturbations grow exponentially provided

$$0 < \delta < \sqrt{2}ka.$$

It is important to note that the instability is controlled by the wave-number k and the amplitude a of the carrier wave (larger values of ka allow more unstable 'modes' δ, thus enhancing the instability).

One practical implication of the Benjamin–Feir instability is the disintegration of periodic wave-trains on sufficiently deep water, as demonstrated by their wave tank experiments (see, e.g., [4]). However, it was experimentally observed [9] that the periodic wave need not always disintegrate (under certain circumstances the instability may lead to a *Fermi–Pasta–Ulam-type recurrence*). This means that the Benjamin–Feir instability saturates and that the initial wave-form is (approximately) regained after a while.

This whole process is best understood by investigating the normalized non-linear **Schrödinger equation**,

$$iA_t + A_{xx} + 2\left|A\right|^2 A = 0,$$

which describes the evolution of weakly non-linear wave envelopes on deep water, among other things (for a general introduction see [3]). This equation also has the distinction that it is completely integrable and that it allows **soliton** solutions. H.C. Yuen and B.M. Lake [9] observed soliton interactions in wave tank experiments, which demonstrates that the non-linear Schrödinger equation provides a qualitatively satisfactorily description of the long-time evolution of wave packets.

The non-linear Schrödinger equation also exhibits the Benjamin–Feir instability and was used in [8] to study